Symmetries in QFTs and applications

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Solving Quantum Field Theories

UV ← microscopic description
- Quantum Chromodynamics
- (Lattice) quantum many-body systems
- Conformal field theories (CFT_{UV})

Renormalization Group (RG) flow

IR ← macroscopic description
- Nonlinear σ-model (chiral Lagrangian)
- Fermi liquid, spin liquid (TQFT), ...
- CFT_{IR}.

This is a very tough problem!

Power of Symmetry

Sometimes, we can know about low-energy dynamics

using Symmetry **without solving microscopic Hamiltonian.**

e.g. In '60s, people didn't know about Quantum Chromodynamics (QCD).

fundamental theory of strong interaction.

current algebra (chiral effective Lagrangian)

$\Longrightarrow$ successful description of low-energy properties of strong interaction

Why this was possible?

Universality due to SSB of chiral symmetry.

Continuous Symmetry in QFT

Noether : If classical action $S[\phi]$ is invariant under continuous transformation,

then there is a conserved current J^μ :

$$\partial_\mu J^\mu = 0.$$

$$\Downarrow$$

In QFT, this becomes Ward-Takahashi identity :

$$\langle \partial_\mu J^\mu(x)\ \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \sum_i \delta(x-x_i)\ \langle \mathcal{O}_1(x_1)\cdots \delta\mathcal{O}_i(x_i)\cdots\mathcal{O}_n(x_n)\rangle.$$

$$\Downarrow$$

Various theorems related to symmetry.

(such as Nambu-Goldstone theorem, 't Hooft anomaly matching, etc.)

Generalization of Symmetry

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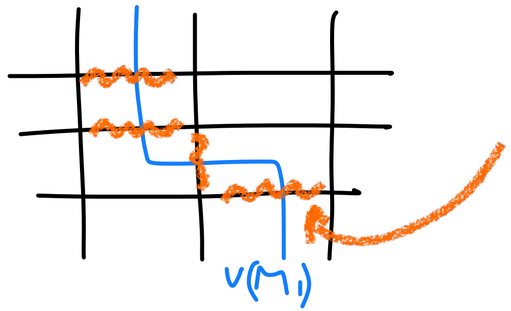
Generalization of Ward-Takahashi identity

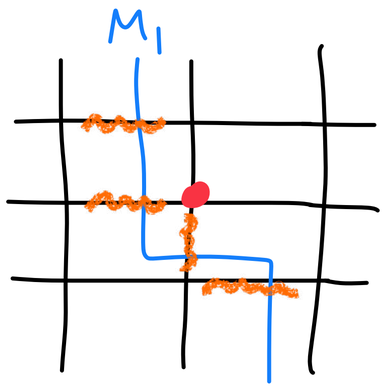
Example WT-type identity for $\mathbb{Z}_2$ symmetry of Ising model

$$H_{\text{Ising}} = J \sum_{\langle x, x' \rangle : \text{nearest neighbors}} s(x)\, s(x') \qquad (s(x) = \pm 1)$$

$V(M_1)$: defect operator

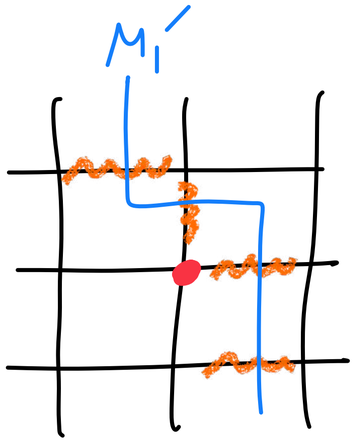


For these $\langle x, x' \rangle$,
change $J \to -J$ in the Hamiltonian.



$\mathbb{Z}_2$ operation for $s(x)$ at ● :

$s \to -s$ $\Longrightarrow$



WT-type identity

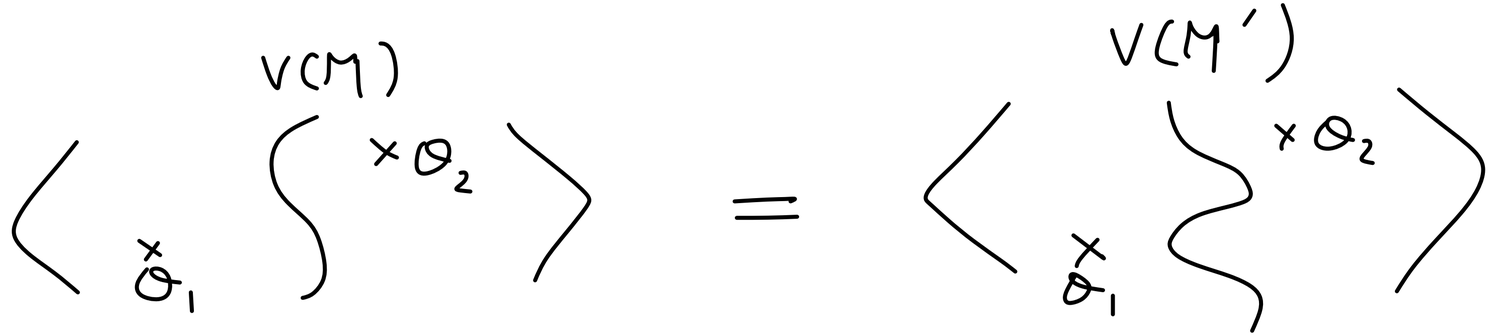
$$\langle V(M_1) \cdots \rangle = \langle V(M_1') \cdots \rangle .$$

Modern Definition of (Generalized) Symmetry

Take-home message

> **Symmetry = Topological defect operators**

Topological



$\Longleftrightarrow$ Conservation Law

(Continuous sym : $V(M) = e^{i\alpha Q(M)} = e^{i\alpha \int_M *(J_\mu dx^\mu)}$.

$\mathbb{Z}_2$ of Ising : $V(M) =$ J in H is replaced by $-J$ when $\langle x, x' \rangle$ crosses M.)

Ordinary Symmetry in Modern Viewpoints

d-dim. QFT has a global symmetry G.

$\overset{\text{def}}{\Longleftrightarrow}$

- $\exists\, V_g(M_{d-1})$: topological codim-1 defect operator for each $g \in G$

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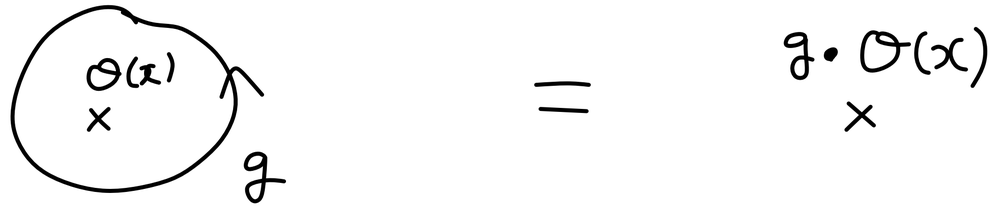
- 

(Valid for both continuous and discrete symmetries)

Various generalizations

- **p-form symmetry** (Gaiotto, Kapustin, Seiberg, Willet '14)

  Topological defects have codim-(p+1): $V_g(M_{d-p-1})$.

  (* Esp., 1-form symmetry generalizes the center sym. in gauge theories.)

- **n-group symmetry** (Sharpe '15, Cordova, Dumitrescu, Intriligator '17, YT, Ünsal '19 ...)

  ≈ Mixture of 0-, 1-, ..., (n-1)-form symmetries.

- **non-invertible symmetry** (Bhardwaj, Tachikawa '17, ... in 2d QFTs.
  '21 ~ higher dim: Nguyen, YT, Ünsal, Koide, Nagoya, Yamaguchi, Choi. Cordova, Hsin, Lam, Shao, Kaidi, Ohmori. Zheng, ...)

  Transformation rule does not form a group

Application : Phase diagram of Fradkin-Shenker's model

Fradkin - Shenker's (non-) complementarity.

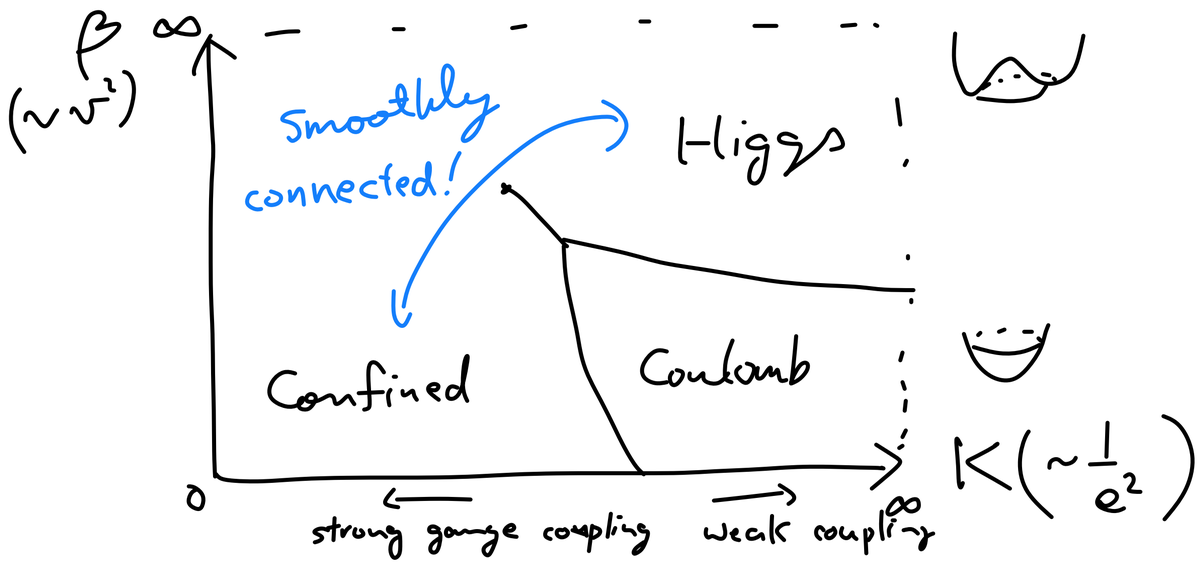
Consider the compact $U(1)$ gauge theory coupled to charge-q scalar ($q=1,2,3,\dots$)

$$S = \frac{1}{2e^2}\int da \wedge *da + \int\left\{\left|(\partial_\mu + i q a_\mu)\phi\right|^2 + g(|\phi|^2 - v^2)^2\right\}.$$

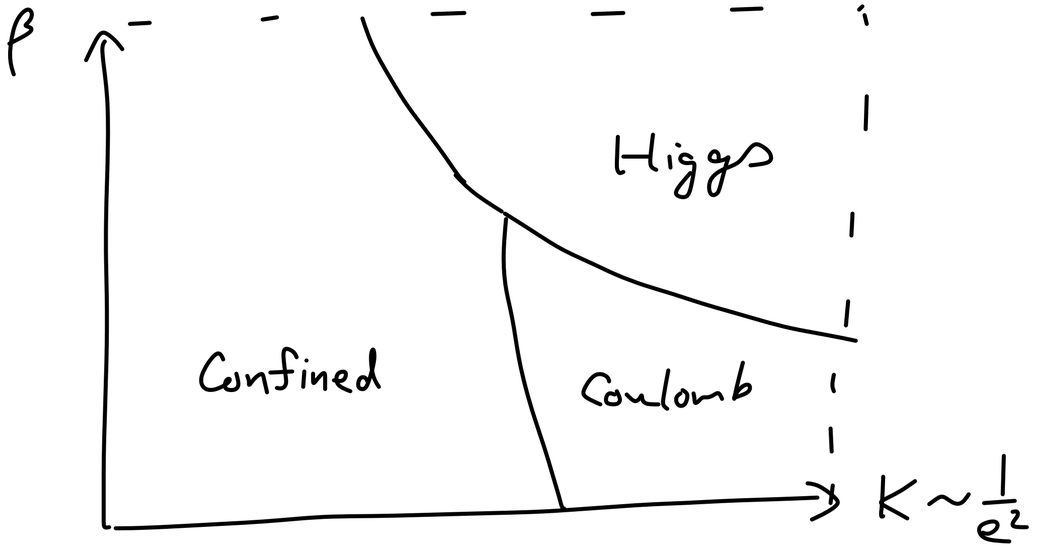
(In Fradkin, Shenker's paper ('79), the lattice version is considered, $(\phi \leftrightarrow e^{i\theta})$

$$S = \beta \sum_i \cos(\partial_\mu \theta + q a_\mu) + K \sum_\square \cos(f_{\mu\nu}) \; . \quad)$$

$\underline{q=1}$ (Confined & Higgs phases are the same)



$\underline{q \geq 2}$ (They are different)



Fradkin - Shenker revisited (Application of 1-form symmetry)

They consider charge-q U(1)-Higgs model on a lattice

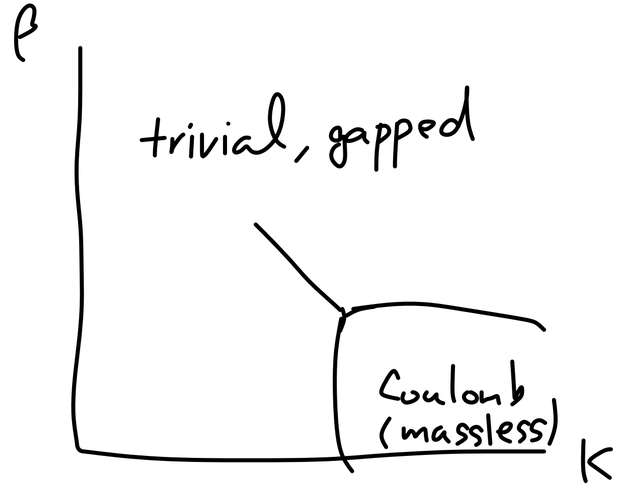
$$S = \beta \sum_{i,\mu} \cos(\partial_\mu \theta + q A_\mu) + K \sum_{\square} \cos(f_{\mu\nu})$$

$$\left(\leftrightarrow \quad S = \frac{1}{2e^2} \int |dA|^2 + \int \left\{ |(\partial_\mu + i q A_\mu)\phi|^2 + \text{W} \right\} + \text{monopoles} \right).$$

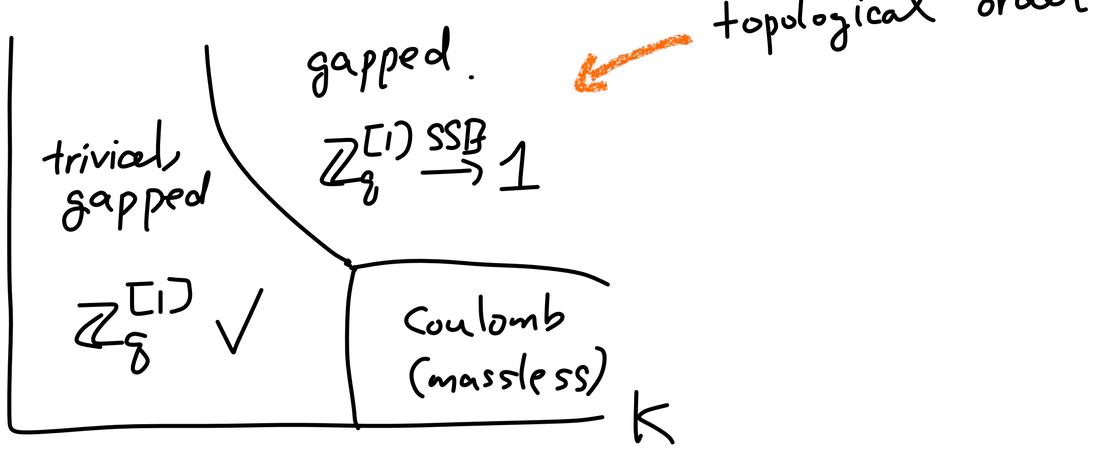
$U(1)_E^{[1]}$ explicit $\longrightarrow$ $\mathbb{Z}_q^{[1]}$

$U(1)_M^{[1]}$ explicit $\longrightarrow$ X

$q = 1$ (No symmetry)



$q \geq 2$ ($\mathbb{Z}_q^{[1]}$ symmetry)



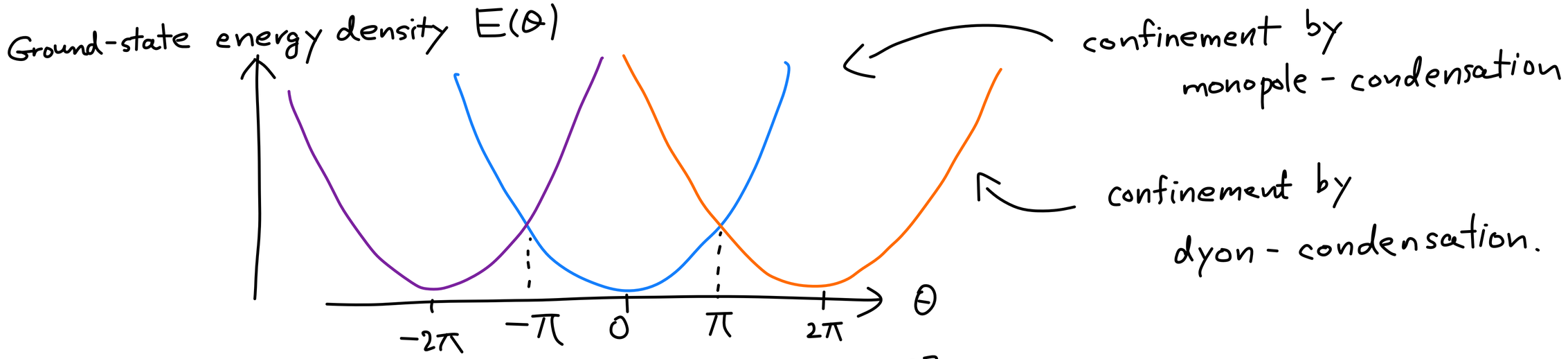
Application : θ-dependence of 4d gauge theories

YM theory at finite θ.

4d gauge theory has two renormalizable terms:

$$\underbrace{\frac{1}{g^2}\int F_{\mu\nu}F_{\mu\nu}}_{\text{kinetic term}} + \underbrace{i\theta\int \varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}}_{\text{topological term } (= \text{instanton number})}.$$

When confinement occurs at any θ, the conjectured phase diagram is



Q. All confinement phases have unbroken $\mathbb{Z}_N^{[1]}$. Is the phase transition at $\theta=\pi$ accidental?

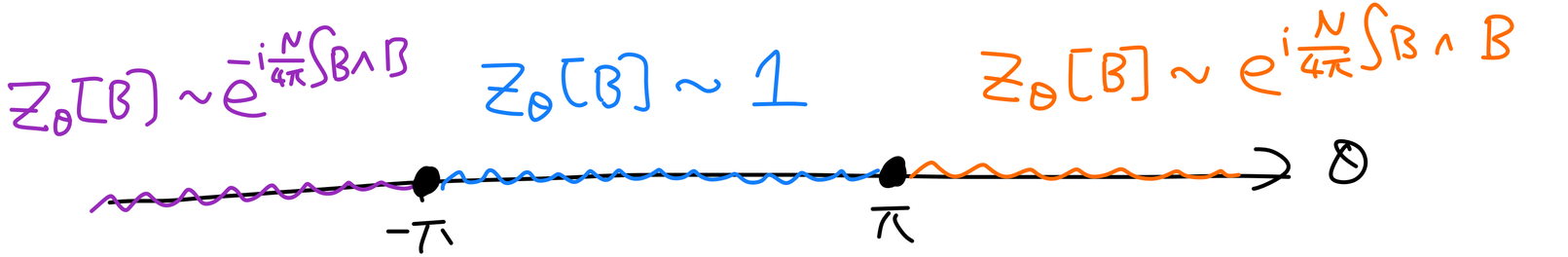
A. These confinement states are different as Symmetry-Protected Topological (SPT) states.

$\Rightarrow$ Phase transition is mandatory. (Gaiotto, Kapustin, Komargodski, Seiberg '17)

B : $\mathbb{Z}_N$ 2-form gauge field (= Background gauge field for $\mathbb{Z}_N^{[1]}$)

$$Z_{\theta+2\pi}[B] = e^{i\frac{N}{4\pi}\int B\wedge B} \times Z_\theta[B].$$

↑ 2π-periodicity of θ is violated by a local counterterm of B.

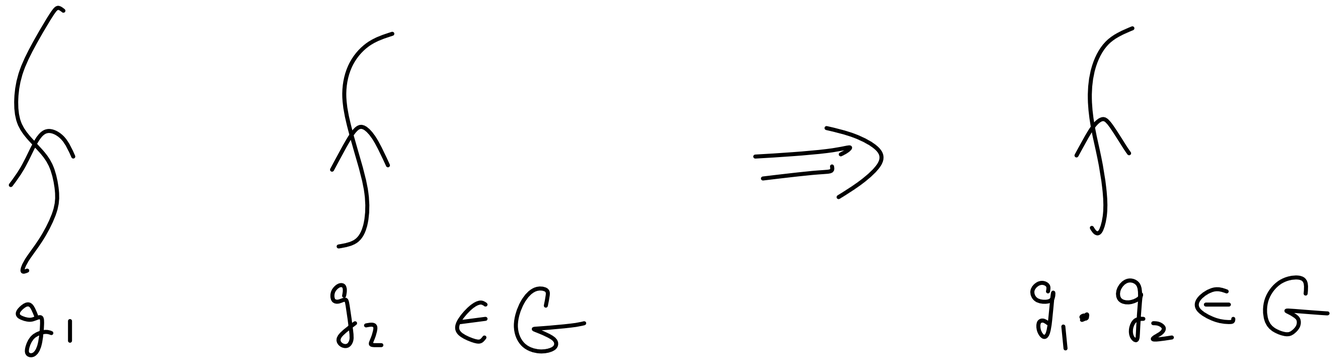


(→ For extension to the case with fundamental quarks, see Kikuchi, YT '17, Shimizu, Yonekura '17, etc.)

Application : Duality Symmetry as Noninvertible Symmetry

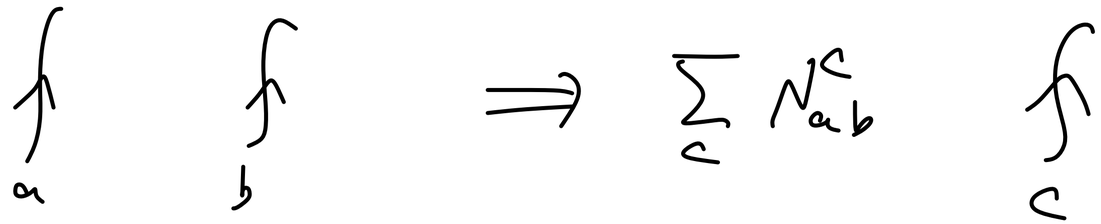
Noninvertible Symmetry

Usually, symmetry forms a group:

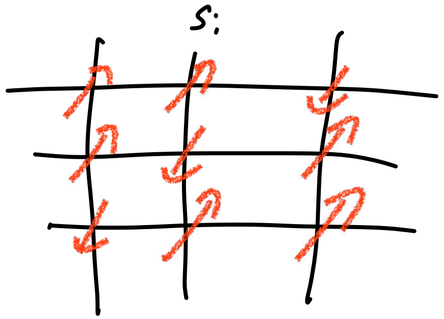


Is this necessary?

More general fusion rule $\Rightarrow$ "noninvertible" (or "categorical") symmetry

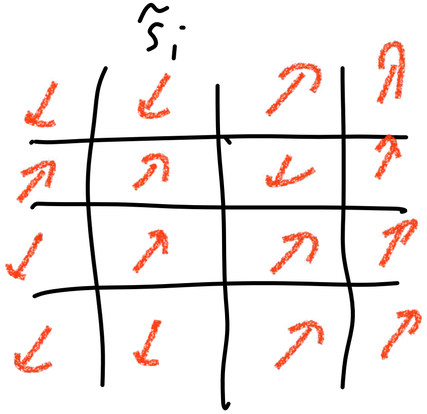


Kramers - Wannier duality of 2d Ising model



$e^{J \sum s_i s_j}$

Dual



$e^{\tilde{J} \sum \tilde{s}_i \tilde{s}_j}$

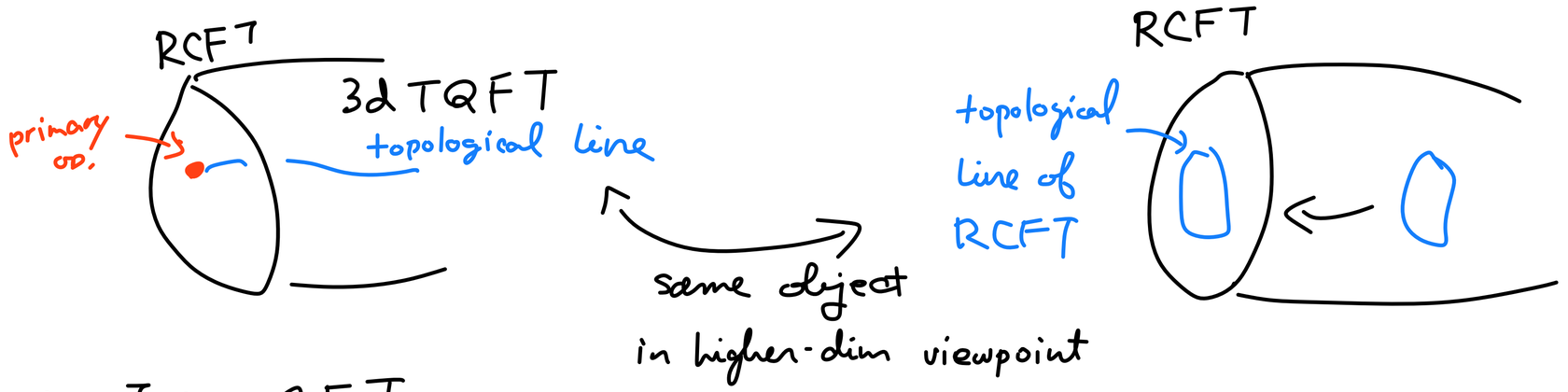
This exchanges the original spin & dual spin.

$$s(x) \longleftrightarrow \tilde{s}(x).$$

Not mutually local

What's this trans?

A. Non-invertible symmetry! (← described by fusion category)



(cf. Verlinde '88)

2d Ising CFT

Primaries : $\underset{\text{trivial}}{1}$, $\underset{\text{energy}}{\varepsilon}$, $\underset{\text{spin}}{\sigma}$

Operator Product Expansion

$$\begin{cases} \sigma \times \sigma = 1 + \varepsilon \\ \sigma \times \varepsilon = \sigma \\ \varepsilon \times \varepsilon = 1 \end{cases}$$

Topological line : 1 , $\underset{\mathbb{Z}_2 \text{ sym}}{\eta}$, $\underset{\text{KW duality}}{D}$

$$\begin{cases} D \times D = 1 + \eta \\ D \times \eta = D \\ \eta \times \eta = 1 \end{cases}$$

(Lattice construction : Aasen, Fendley, Mong $\overset{\text{extension to 4d}}{\Longrightarrow}$ Koide, Nagoya, Yamaguchi; Choi, Cordova, et al. ; Kaidi, Ohmori, Zheng)

Application : Solitonic symmetry beyond homotopy

[Based on my work with Shi Chen @ U. Tokyo
It will appear tomorrow's morning on arXiv.]

Nonlinear σ-model $\quad \sigma : M_d \longrightarrow X$

spacetime $\quad$ target space

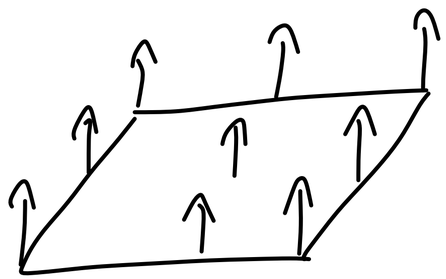
$$Z = \int \mathcal{D}\sigma \, \exp(-S[\sigma]), \quad S[\sigma] = \int d^d x \; g^{\mu\nu}(\sigma)\, \partial_\mu \sigma \partial_\nu \sigma .$$

$X = \mathbb{C}P^1 \; (= S^2)$: classical ferromagnet, quantum anti-ferromagnet, etc.

$X = SU(N_f)$: Quantum ChromoDynamics (QCD).

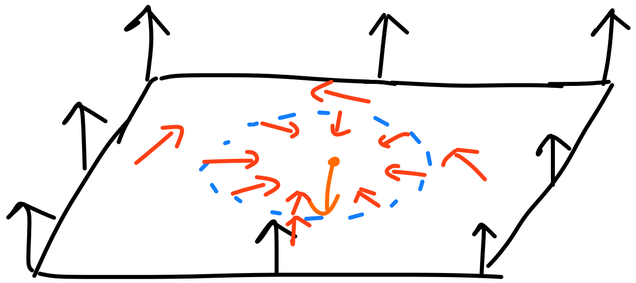
Solitons : stable nonperturbative excitations due to topology

↖ (conventionally) classified by homotopy groups



trivial

$0 \in \pi_2(\mathbb{C}P^1)$



non-trivial

(hedgehog configuration)

$1 \in \pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}$

Claim There is a more rich structure for solitonic symmetries.

4d $\mathbb{C}P^1$ σ-model

There are two kinds of solitons:

$\pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}$ $\longleftrightarrow$ stable vortex

$\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ $\longleftrightarrow$ stable particle (Hopfion)

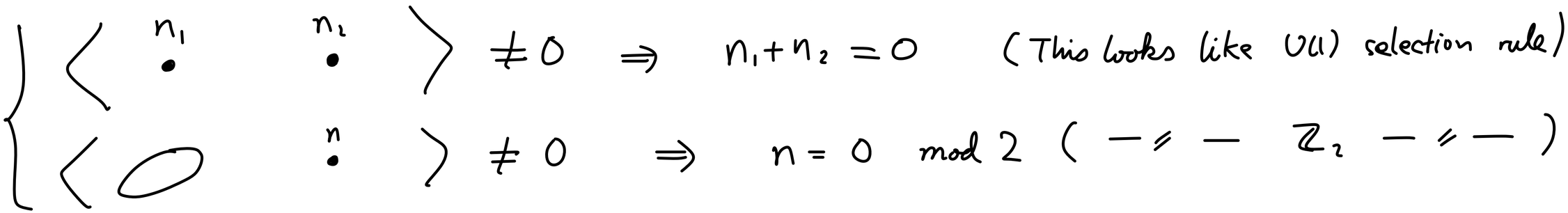
Hopf fibration $S^1 \to S^3 \to S^2$

This homotopy argument has been understood as the presence of solitonic U(1) symmetries.

$\pi_2(\mathbb{C}P^1) \simeq \mathbb{Z}$ $\longleftrightarrow$ U(1) 1-form symmetry

$\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z}$ $\longleftrightarrow$ U(1) ~~0-form~~ symmetry

Non-invertible symmetry $\supset \mathbb{Z}_2$ subgroup



(↑ This follows from the result of Pontryagin '41)

Topological operators

ω_{S^2} ← volume 2-form of unit S^2

$$\omega_{S^2} =: \frac{da}{2\pi} \qquad (a : U(1) \text{ gauge field})$$

$\mathbb{Z}_2$ symmetry generator : ← π × Hopf inv.

$$\mathcal{H}_\pi(M_3) = \exp\left(i \int_{M_3} \frac{a\,da}{4\pi}\right)$$

$$\widetilde{\Omega}^{\text{Spin}}_3(\mathbb{CP}^1) \simeq \mathbb{Z}_2$$

This generates "π rotation" for Hopf charge.

For a fractional rotation $\alpha = \frac{\pi}{N}$, we can construct

$$\mathcal{H}_{\frac{\pi}{N}}(M_3) = \int \mathcal{D}b \exp\left(i \frac{N}{4\pi} \int_{M_3} b \wedge db + i \frac{1}{2\pi} \int_{M_3} b \wedge da\right).$$

↳ Non-invertible Hopf number symmetry.

(Generator is a partition function of 3d topological QFTs)

Summary

Take-home message

Symmetry = Topological defect operators

$\Rightarrow$ New aspects of strongly-coupled QFTs.